Krull, Gelfand-Kirillov and Filter Dimensions of Simple Affine Algebras

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*T Talks/lectbonn
Plan


2. Filter Dimension.

3. Analog of the Inequality of Bernstein for Simple Affine Algebras.

4. Inequality between Krull, Gelfand-Kirillov and Filter Dimensions for Simple Affine Algebras. Applications to $\mathcal{D}$-modules.
1. Gelfand-Kirillov Dimension

module=left module

$K$ is a field of char 0

$\mathbb{N}$ and $\mathbb{R}$ are sets of natural and real numbers

**Definition.** For a function $f : \mathbb{N} \to \mathbb{N}$, the real number or $\infty$ defined as

$\gamma(f) := \inf \{ r \in \mathbb{R} : f(i) \leq i^r \text{ for suff. large } i \gg 0 \}$

is called the degree (or growth) of $f$.

For functions $f, g : \mathbb{N} \to \mathbb{N}$:

$\gamma(f + g) \leq \max\{\gamma(f), \gamma(g)\}$,

$\gamma(fg) \leq \gamma(f) + \gamma(g)$,

$\gamma(f \circ g) \leq \gamma(f)\gamma(g)$,

where $f \circ g$ is the composition of the functions.
Let $A$ be an affine (≡ finitely generated) algebra with generators $x_1, \ldots, x_n$. Then $A$ is equipped with a standard finite dimensional filtration

$$A = \bigcup_{i \geq 0} A_i, \quad A_0 = K,$$

$$A_1 = K + \sum_{i=1}^{n} Kx_i, \quad A_i := A_1^i, \ i \geq 2.$$  

Let $M$ be a finitely generated $A$-module and $M_0$ be a finite dimensional generating subspace of $M$, $M = AM_0$. The module $M$ has a finite dimensional filtration

$$M = \bigcup_{i \geq 0} M_i, \quad M_i = A_iM_0.$$  

**Definition (Gelfand-Kirillov, 1966).** The **Gelfand-Kirillov** dimension of the $A$-module $M$:

$$\text{GK}(M) := \gamma(i \to \dim M_i).$$

The **Gelfand-Kirillov** dimension of the algebra $A$:

$$\text{GK}(A) := \gamma(i \to \dim A_i).$$
GK(M) and GK(A) do not depend on the choice of the filtrations.

**Example.** Let $P_n = K[X_1, \ldots, X_n]$ be the polynomial ring in $n$ indeterminates.

$$P_n = \cup_{i \geq 0} P_{n,i}, \quad P_{n,0} = K, \quad P_{n,1} = K + \sum_{i=1}^{n} KX_i,$$

$$P_{n,i} = \sum \{KX^\alpha \mid |\alpha| \leq i\}, \quad X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

- $\dim P_{n,i} = \binom{n+i}{n}$
  $$= (i+n)(i+n-1) \cdots (i+1)/n! = \frac{i^n}{n!} + \cdots.$$

- $\text{GK}(P_n) = n.$

The $n$'th **Weyl algebra**

$$A_n = K < X_1, \ldots, X_n, \partial_1, \ldots, \partial_n >$$
defining relations:
\[
\partial_i X_j - X_j \partial_i = \delta_{ij}, \text{ the Kronecker delta},
\]
\[
X_i X_j - X_j X_i = \partial_i \partial_j - \partial_j \partial_i = 0, \ i, j = 1, \ldots, n.
\]
The algebra $A_n$ is a simple Noetherian infinite dimensional algebra canonically isomorphic to the ring of differential operators with polynomial coefficients
\[
A_n \simeq K[X_1, \ldots, X_n, \partial/\partial X_1, \ldots, \partial/\partial X_n],
\]
\[
X_i \leftrightarrow X_i, \ \partial_i \leftrightarrow \partial_i/\partial X_i, \ i = 1, \ldots, n.
\]

- $\{X^\alpha \partial^\beta\}$ is a $K$-basis of $A_n$.

- A filtration: $A_n = \bigcup_{i \geq 0} A_{n,i},$
\[
A_{n,i} = \sum \{K X^\alpha \partial^\beta, |\alpha| + |\beta| \leq i\},
\]
\[
\dim A_{n,i} = \binom{2n + i}{2n} = \frac{i^{2n}}{(2n)!} + \cdots.
\]

- $\text{GK}(A_n) = 2n.$
2. Filter Dimension

**Lemma 1.** Let $A$ be a simple affine inf. dim. algebra and let $M \neq 0$ be a f.g. $A$-module. Then $\dim M = \infty$, hence $\text{GK}(M) \geq 1$.

**Proof.** The alg. $A$ is simple, so the nonzero algebra homomorphism

$$A \to \text{Hom}_K(M, M), a \mapsto (m \mapsto am),$$

is injective, so $\infty = \dim A \leq \dim \text{Hom}_K(M, M)$ and $\dim M = \infty$.

**Theorem 2.** (The inequality of Bernstein, 1972). For a nonzero finitely generated module $M$ over the Weyl algebra $A_n$,

$$\text{GK}(M) \geq n.$$  

Let $X$ be a smooth irreducible algebraic variety of dimension $n$. Let $\mathcal{D}(X)$ be the ring of differential operators on $X$.  

7
Example. \( X = K^n, \mathcal{D}(K^n) = A_n; \)

\[ X = S^n := \{(x_i) \in K^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}, \mathcal{D}(S^n). \]

The alg. \( \mathcal{D}(X) \) is a simple affine Noetherian inf. dim. algebra with \( \text{GK}(\mathcal{D}(X)) = 2n. \)

**Theorem 3.** (McConnell-Robson). For a nonzero finitely generated \( \mathcal{D}(X) \)-module \( M, \)

\[ \text{GK}(M) \geq n. \]

**Question.** How to find (estimate) the number \( i_A := \inf \{\text{GK}(M), \ M \text{ is a nonzero finitely generated } A \text{ – module}\}? \)

To find an approximation for \( i_A \) was a main motivation for introducing the filter dimension.
Let $A$ be a simple affine algebra with the filtration $F = \{A_i\}$, $A = \bigcup_{i \geq 0} A_i$. Define the return function $\nu_F : \mathbb{N} \to \mathbb{N}$ of $A$:

$$\nu_F(i) := \inf \{ j \in \mathbb{N} : A_j a A_j \ni 1 \text{ for all } 0 \neq a \in A_i \},$$

where $A_j a A_j$ is the subspace of $A$ generated by the products $xay$, for all $x, y \in A_j$.

**Definition.** (B., 1996). The filter dimension of $A$:

$$\text{fdim } A := \gamma(\nu_F).$$

The filter dimension does not depend on the choice of $F$. 
3. Analog of the Inequality of Bernstein for Simple Affine Algebras

**Theorem 4.** (B., 1996). Let $A$ be a simple affine infinite dimensional algebra.

1. $\text{fdim } A \geq 1/2$.

2. For every nonzero finitely generated $A$–module $M$:

$$
\text{GK}(M) \geq \frac{\text{GK}(A)}{\text{fdim}(A) + \max\{\text{fdim}(A), 1\}}.
$$

**Proof.** 2. Let $A = K < x_1, \ldots, x_n > = \bigcup_{i \geq 0} A_i$ and let $F = \{A_i\}$ be the filtration of $A$. 

Let $M_0$ be a fin. dim. gen. subspace of the $A$-module $M$:

$$M = \cup_{i \geq 0} M_i, \quad M_i = A_i M_0, \quad i \geq 0.$$ 

It follows from the definition of the return function $\nu = \nu_F$ of $A$ that, for every $0 \neq a \in A_i$, $1 \in A_{\nu(i)} a A_{\nu(i)}$. Now,

$$M_0 = 1 M_0 \subseteq A_{\nu(i)} a A_{\nu(i)} M_0 \subseteq A_{\nu(i)} a M_{\nu(i)},$$

so the linear map

$$A_i \to \text{Hom}_K(M_{\nu(i)}, M_{\nu(i)+i}), \quad a \mapsto (m \mapsto a m),$$

is injective, hence

$$\dim A_i \leq \dim \text{Hom}_K(M_{\nu(i)}, M_{\nu(i)+i})$$

$$= \dim M_{\nu(i)} \dim M_{\nu(i)+i}.$$ 

Using the elementary properties of the degree, we have

$$\text{GK}(A) := \gamma(\dim A_i) \leq \gamma(\dim M_{\nu(i)} \dim M_{\nu(i)+i})$$
\[
\leq \gamma(\dim M_{\nu(i)}) + \gamma(\dim M_{\nu(i)+i}) \\
\leq \gamma(\dim M_i)\gamma(\nu) \\
+ \gamma(\dim M_i) \max\{\gamma(\nu), 1 = \gamma(i \to i)\} \\
= \text{GK}(M)(\text{fdim } A + \max\{\text{fdim } A, 1\}),
\]

since

\[
\text{GK}(M) = \gamma(\dim M_i) \quad \text{and} \quad \text{fdim } A = \gamma(\nu).
\]
Theorem 5. (B., 1998). Let $\mathcal{D}(X)$ be the ring of differential operators on a smooth irreducible algebraic variety $X$ of dimension $n$. The filter dimension

$$\text{fdim} \mathcal{D}(X) = 1.$$  

- (McConnell-Robson). For a nonzero finitely generated $\mathcal{D}(X)$-module $M$,

$$\text{GK}(M) \geq n.$$  

Proof.

$$\text{GK}(M) \geq \frac{\text{GK}(\mathcal{D}(X))}{\text{fdim}(\mathcal{D}(X)) + \max\{\text{fdim}(\mathcal{D}(X)), 1\}}$$

$$= \frac{2n}{1 + \max\{1, 1\}} = \frac{2n}{2} = n.$$
4. Inequality between Krull, Gelfand-Kirillov and Filter Dimensions for Simple Affine Algebras. Applications to $\mathcal{D}$-modules

K.dim, the **Krull dimension** (in the sense of Rentschler-Gabriel, 1967)

**Theorem 6** (Rentschler-Gabriel, 1967) *Let $A_n$ be the Weyl algebra. Then K.dim $A_n = n$.**

**Theorem 7** (McConnell-Robson) *Let $X$ be a smooth irreducible algebraic variety of dim $n$. Then K.dim $\mathcal{D}(X) = n$.**
**Definition.** An algebra \( S \) is called **finitely partitive** if, given any fin. gen. \( S \)-module \( M \), there is an integer \( n > 0 \) s. t. for every chain of submodules

\[
M = M_0 \supset M_1 \supset \ldots \supset M_m
\]

with \( \text{GK}(M_i / M_{i+1}) = \text{GK}(M) \), one has \( m \leq n \).

**Lemma 8.** \( \mathcal{D}(X) \) is a finitely partitive alg., and for any fin. gen. \( \mathcal{D}(X) \)-module \( M \), \( \text{GK}(M) \) is a natural number.

**Theorem 9** (B., 1998) Let \( A \) be a finitely partitive simple affine algebra with \( \text{GK}(A) < \infty \). Suppose that the Gelfand-Kirillov dimension of every finitely generated \( A \)-module is a natural number. Then, for any nonzero finitely generated \( A \)-module \( M \), the Krull dimension

\[
\text{K.dim}(M) \leq \text{GK}(M) - \frac{\text{GK}(A)}{\text{fdim}(A) + \max\{\text{fdim}(A), 1\}}.
\]

In particular,

\[
\text{K.dim}(A) \leq \text{GK}(A)(1 - \frac{1}{\text{fdim}(A) + \max\{\text{fdim}(A), 1\}}).
\]
• (McConnell-Robson). $\text{K.dim } \mathcal{D}(X) = n$.

**Proof.** $\text{GK } \mathcal{D}(X) = 2n$ and $\text{fdim } \mathcal{D}(X) = 1$.

By Theorem 9,

$$\text{K.dim } \mathcal{D}(X) \leq 2n \left(1 - \frac{1}{1 + \max\{1,1\}}\right)$$

$$= 2n \left(1 - \frac{1}{2}\right) = \frac{2n}{2} = n.$$ 

$\text{K.dim } \mathcal{D}(X) \geq n$, easy. 

A generalization to AFFINE ALGEBRAS is given in


(Faithful and Schur Dimensions)